Group Theory Third Exam

Date: 24 March 2022

Time: 17:00 – 19:00

Instructions

- To get full points, you must provide complete arguments and computations. You will get no points if you do not explain your answer. Answers like "True", "False", "34", "Yes", "No" will not be accepted.
- While solving a problem, you can use any statement that needs to be proven as a part of another problem even if you did not manage to prove it; e.g. you can use part (a) while solving part (b) even if you did not prove (a).
- Clearly write your name and student number on each page you submit.
- The examination consists of 5 questions. You can score up to 36 points and you get 4 points for free. This way you will score in total between 4 and 40 points.

Problems

1. Let S_{13} be the permutation group on $\{1, 2, 3, \dots, 13\}$ and let $\sigma = (2\ 4\ 7\ 8)(4\ 6\ 8)(5\ 7)(1\ 13) \in S_{13}$ and $\tau = (12\ 11\ 3\ 9\ 10) \in S_{13}$.
   (a) **[2 pt]** Write $\gamma := (\tau \cdot \sigma)^{2024}$ as a product of disjoint cycles.
   (b) **[2 pt]** What is the sign of γ?
   (c) **[3 pt]** Let H be the subgroup of S_{13} generated by σ and τ. Show that H is cyclic and compute the order of H.

2. Let G be a group and $T \subset G$ a subset (not necessarily a subgroup). For every $a \in G$ we define
$$aTa^{-1} = \{ata^{-1} : t \in T\}.$$
   Now define
$$F(T) = \{a \in G : aTa^{-1} = T\}.$$
   (a) **[2 pt]** Show that $F(T)$ is a subgroup in G.
   (b) **[2 pt]** Explain why $\#\{aTa^{-1} : a \in G\} = [G : F(T)]$.
   (c) **[2 pt]** Give an example of a group G and a subset $T \subset G$ for which the corresponding subgroup $F(T)$ is not normal in G.

3. (a) **[2 pt]** Show that the map
$$S_4 \times \mathbb{R}^4 \to \mathbb{R}^4$$
$$(\sigma, (x_1, x_2, x_3, x_4)) \to \sigma \cdot (x_1, x_2, x_3, x_4) := (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$$
   is a group action. (Here $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$)
   (b) **[3 pt]** Is the action in (a) faithful; transitive; or fixed point free? Explain.
   (c) **[2 pt]** Take $r := (2, 4, 4, 2) \in \mathbb{R}^3$. Compute the stabilizer and the orbit of r under the action in (a).

4. (a) **[3 pt]** Let $H := (\mathbb{Z}/15\mathbb{Z})^\times$ and $K := (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z})/\langle(\bar{2}, \bar{2})\rangle$. Find the elementary divisors of the group $H \times K$.
Note: Here $\langle(\bar{2}, \bar{2})\rangle$ denotes the subgroup in $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ generated by the element $(\bar{2}, \bar{2})$.

   (b) **[4 pt]** List, up to isomorphism, all abelian groups of order between 14 and 38 containing an element of order 6 and no element of order 4.

5. **True/False.** Prove the following statements if they are correct and disprove them if they are not.

   (a) **[3 pt]** There is a surjective homomorphism from D_{12} to $(\mathbb{Z}/5\mathbb{Z})^\times \times (\mathbb{Z}/7\mathbb{Z})^\times$.

   (b) **[3 pt]** If a finite group has only 2 conjugacy classes then it has order 2.

   (c) **[3 pt]** Any group of order 15 is cyclic.

GOOD LUCK! ☺